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Recurrence Properties of Lorentz Lattice Gas Cellular Automata

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Recurrence properties of a point particle moving on a regular lattice randomly occupied with scatterers are studied for strictly deterministic, nondeterministic, and purely random scattering rules.

KEY WORDS: Recurrence; percolation; cellular automata; tiling; lattice gas; Lorentz gas; wind-tree model.

1. INTRODUCTION

In the classical Lorentz gas one particle moves in $\mathbf{R}^3$ and collides elastically with randomly placed spheres.[(1)] In the Ehrenfest wind-tree model the particle is elastically scattered by randomly placed diamonds whose diagonals are parallel to the coordinate axes.[(2)] In this paper we consider several models of cellular automata which in a sense can be thought of as the restriction of either of the above models to the square lattice $\mathbf{Z}^2$ or to the regular triangular lattice $\mathbf{T}^2$. Our aim is the rigorous analysis of the recurrence properties of the various models. The recurrence properties change dramatically when the collision rules change from various combinations of deterministic and probabilistic rules. We also consider a related purely random model which is not of cellular automaton type.

2. DESCRIPTION OF THE MODELS AND STATEMENT OF THE RESULTS

The models considered here are mostly lattice versions of Lorentz gas (or wind-tree) models. The first, which was introduced by Ruijgrok and

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Fig. 1. (a) Right mirror. (b) Left mirror.

Cohen,[3] we call the deterministic mirror model (DM). In this model two-sided mirrors, interpreted as scatterers or trees, are placed at the sites of the square lattice $\mathbf{Z}^2$. They can align along either one of the diagonal directions of the lattice, and will be called left or right mirrors, depending on the direction (Fig. 1). The mirrors are placed randomly on $\mathbf{Z}^2$ with respect to the infinite product probability measure μ on the space $\Omega := \{0, L, R\}^{\mathbf{Z}^2}$ determined by the concentrations C_L and C_R of left and right mirrors with the obvious constraint $C := C_L + C_R \leqslant 1$. A (wind) particle with unit speed and four possible directions propagates along the bonds of the lattice and is reflected by the scatterers (Fig. 2). Thus, the DM model is a deterministic cellular automaton. If C_R or C_L equals 0, then the path of the particle travels monotonically toward ∞ (Fig. 3).

Theorem 1. For the DM model on the $\mathbf{Z}^2$ lattice, if $C = 1$, $C_R > 0$, and $C_L > 0$, then with probability 1 orbits are periodic.

Grimmett has proven this theorem in the case $C_R = C_L = 1/2$.[4] He conjectures that for $C < 1$ the probability of infinite orbits is positive for the case $C_R = C_L$. Cohen, Kong, Ruijgrok, and Ziff[3,5–9] have extensively studied this model. Their latest simulations indicate that this may not be the case.[8,9]

The DM model, in the case $C = 1$, $C_R > 0$, and $C_L > 0$, is equivalent to the model of randomly tiling the plane with "Truchet tiles," shown in

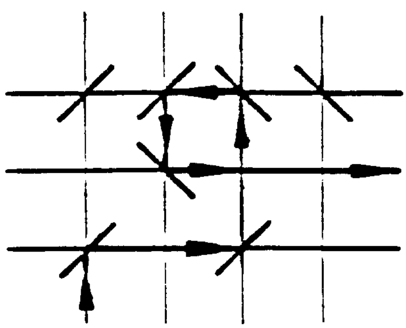

Fig. 2.

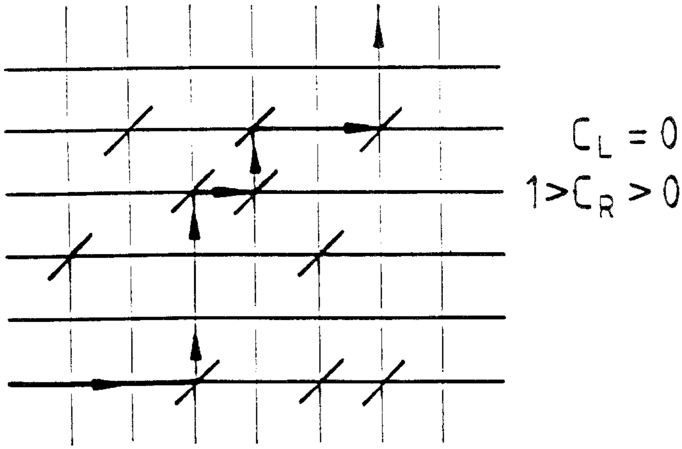

Fig. 3.

Fig. 4. To get such a tiling from the DM model, trace out all paths on a configuration, but curve the path as it hits the mirror. Finally, erase all the mirrors. This model was introduced by Pickover[10,11] in connection with visual tests to check if 0–1 sequences are random or not. Roux *et al.*[12] have also independently introduced this model. Various authors have studied relations to fractal dimension of "hulls" of percolation clusters, formations of polymers, smart kinetic walks, transport properties, etc.[13–16]

The next model we consider has a different scatering rule. Instead of left and right mirrors on $\mathbf{Z}^2$, we have rotators (revolving doors) that revolve clockwise and counterclockwise. These rotators always turn the

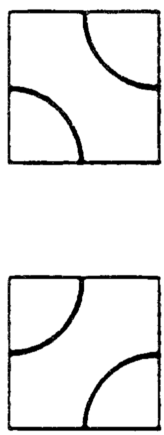

Fig. 4. Truchet tiles.

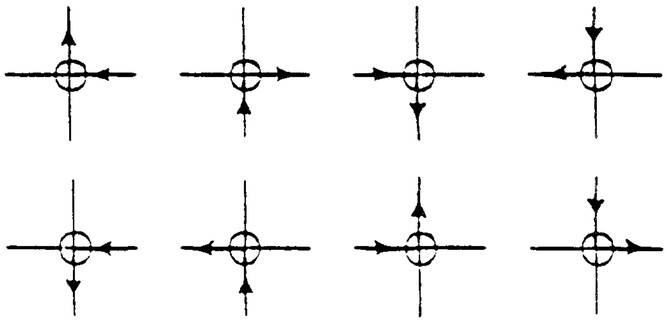

Fig. 5. (a) Right rotator. (b) Left rotator.

particle 90° to the left and right, respectively (Fig. 5). Again we consider an independent measure on the probability space $\Omega := \{0, L, R\}^{\mathbf{Z}^2}$. This model, which we call the deterministic rotator model (DR), was introduced by Gunn and Ortuño.[17]

Theorem 2. For the DR model on $\mathbf{Z}^2$:

(i) There exists $p_c \in (0.5, 1)$ such that orbits are periodic with probability 1 if $C_L > p_c$ or $C_R > p_c$.

(ii) If $C = 1$, then orbits are periodic with probability 1.

The constant p_c is the critical parameter for site percolation on $\mathbf{Z}^2$. Numerical studies show that $p_c \approx 0.59$. Kong and Cohen[7] have noticed behavior similar to this in their simulations.

Kong and Cohen[6] have defined models on the regular equilateral triangular lattice $\mathbf{T}^2$ which are analogous to the DM and DR models on $\mathbf{Z}^2$. For the DM model the two types of mirrors, left and right (Fig. 6), are placed on the sites of the lattice. For the DR model the scatterers turn the particle 60° clockwise or counterclockwise (Fig. 7). We again consider

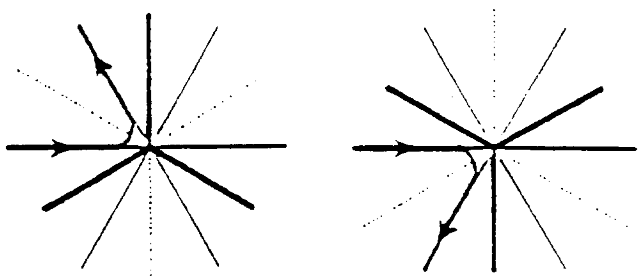

Fig. 6. (a) Right mirror. (b) Left mirror.

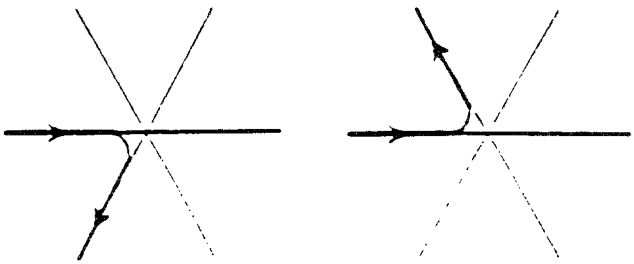

Fig. 7. (a) Right rotator. (b) Left rotator.

an independent probability measure μ on the space $\Omega := \{0, L, R\}^{\mathbf{T}^2}$. As before, a particle with unit speed propagates along the bonds of the lattice and is scattered by the scatterers.

Theorem 3. For the DM and DR models on the $\mathbf{T}^2$ lattice if $C_L \geqslant 1/2$ or $C_R \geqslant 1/2$, then with probability 1 orbits are periodic.

Corollary. If $C = 1$, then with probability 1 orbits are periodic.

For any of the models considered, let $P(\mathbf{r}, t)$ be the probability density to find the particle at the position $\mathbf{r} \in \mathbf{Z}^2$ (resp. $\mathbf{T}^2$) at time t, when it started at the origin at time $t = 0$. Here we choose each of the original four directions (resp. six) with equal probability. For a normal diffusion process, $P(\mathbf{r}, t)$ is asymptotically Gaussian, i.e., there exists a constant D such that

$$P(\mathbf{r}, t) \sim \frac{1}{4\pi Dt} \exp\left(\frac{-|r|^2}{4\pi Dt}\right) \tag{1}$$

The mean square displacement of the particle is then

$$\Delta(t) = \int |\mathbf{r}|^2 P(\mathbf{r}, t)\, d\mathbf{r} \sim 4Dt$$

A diffusion is called non-Gaussian if the asymptotic distribution $P(\mathbf{r}, t)$ is not of the form (1).

Theorem 4. The diffusions for the DM and DR models on $\mathbf{Z}^2$ and $\mathbf{T}^2$ are non-Gaussian.

The simulations of Kong and Cohen[5] indicate that the diffusion is anomalous, that is, it is non-Gaussian, but the mean-square displacement is still proportional to t. Theorem 4 also holds for the natural generalizations of these models to any regular finite-dimensional lattice (for example, the $\mathbf{Z}^n$ lattice for any $n \geqslant 2$).

In $\mathbf{Z}^2$ label the four edges coming to a vertex 0, 1, 2, 3 so that on a clock the edge corresponds to $3i$ o'clock. A general scattering rule is given by a function $f: \{0, 1, 2, 3\} \to \{0, 1, 2, 3\}$. There are 256 such general scattering rules. For example, a left mirror in the DM model is given by the rule $f(i) = 1 - i \bmod 4$. Two rules f and g are rotationally equivalent ($f \sim_r g$) if $\exists k$ such that $f(i) + k \bmod 4 \equiv g(i + k \bmod 4)$. A set of general scattering rules is called rotationally symmetric if it contains only complete rotational symmetry classes (S is complete if $\forall g \in S$ if $f \sim_r g$, then $f \in S$). For example, the set consisting of a left and a right mirror is a complete rotational symmetry class. The set $\{\mathrm{Id}\}$, where $\mathrm{Id}(i) := i + 2 \bmod 4$ (the straight-ahead mapping), is called the trivial rotationally symmetric class. The same can be defined for $\mathbf{T}^2$, except now there are six edges, $f \sim_r g$ if $\exists k$ such that $f(i) + k \bmod 6 \equiv g(i + k \bmod 6)$ and $\mathrm{Id}(i) := i + 3 \bmod 6$ is the trivial rotational symmetry class. For $\mathbf{Z}^n$ there are $(2n)^{2n}$ general scattering rules. Let $e_m := (0,..., 0, 1, 0,..., 0)$, where the 1 is in the mth spot. Let $A_{m,n}$ be the plane through the origin spanned by the vectors e_m and e_n. Two general scattering rules f and g are called rotationally equivalent if $\exists m$, n $(m \neq n)$ and k such that (assuming for convenience that the four edges in the plane $A_{m,n}$ are numbered 0, 1, 2, 3 in clockwise order) $f(i) + \delta(i) \bmod 4 \equiv g(i + \delta(f(i)) \bmod 4)$, where $\delta(j) = 0$ if $j \notin \{0, 1, 2, 3\}$ and $\delta(j) = k$ if $j \in \{0, 1, 2, 3\}$. Again Id, the straight ahead mapping, is called the trivial rotational symmetry class. Suppose Ω is a configuration space of (some) generalized scattering rules on $\mathbf{Z}^2$, $\mathbf{T}^2$, or $\mathbf{Z}^n$ and μ is an independent probability measure on Ω.

Theorem 5. If μ gives positive measure to each element of any nontrivial rotationally symmetric class on $\mathbf{T}^2$ or $\mathbf{Z}^n$ $(n \geqslant 2)$, then the diffusion is non-Gaussian.

The condition of rotational symmetry, although a natural physical condition to impose, is far from being necessary for the diffusion to be non-Gaussian. For example, the diffusion is non-Gaussian for the pair $\{f, g\}$ given by $f(0) = 0$, $f(1) = 2$, $f(2) = 3$, $f(3) = 1$, $g(0) = 1$, $g(1) = 3$, $g(2) = 2$, and $g(3) = 0$. Theorem 5 can be extended in the obvious way to any regular finite-dimensional lattice.

The next model is the flipping mirror model (FM). The model is identical to the DM model except in the rules of the propagation of the particle. Fix p such that $0 < p \leqslant 1$. A particle again propagates along the bonds of the lattice and is reflected by the scatterers, but in addition a mirror changes from left to right and vice versa independently with probability p when it is hit by a particle. It is worthwhile to stress that we consider the motion of a single particle in the lattice. For the DM model there is no difference from the case when there are many moving particles in the lattice.

However, for the FM model this difference is essential because the flipping generates some special type of interaction between particles. For $p \in (0, 1)$ the FM model is a stochastic cellular automaton. For $p = 1$ it is a deterministic cellular automaton. In the spacial case $p = 1$ the FM model was introduced by Ruijgrok and Cohen[3] on $\mathbf{Z}^2$ and by Kong and Cohen[6] on $\mathbf{T}^2$. E. G. D. Cohen has communicated to us that the case $p \neq 1$, although unpublished, was also considered in their computer experiments and is briefly mentioned in ref. 8. Completely analogously, we can define a flipping rotator model (FR). We say that an orbit σ is unbounded if there is no compact set $L \subset \mathbf{Z}^2$ (resp. $\mathbf{T}^2$) such that $\sigma \subset L$.

Theorem 6. For the FM model on $\mathbf{Z}^2$ all orbits are unbounded and thus no orbit is periodic. For the FR model on $\mathbf{Z}^2$ and the FM and FR models on $\mathbf{T}^2$ if $p = 1$ and $0 < C < 1$, then the probability of the origin being periodic is positive, while for $p = C = 1$ or $p < 1$ or $C = 0$ all orbits are unbounded and thus no orbit is periodic.

Several parts of this theorem were already known; in particular, Kong and Cohen[6] have shown that for the FM model on $\mathbf{T}^2$ if $p = 1$ and $C = 1$, then all orbits are unbounded and in fact all orbits go to infinity in a "linear" way.[6] This works equally well for the FR model on $\mathbf{T}^2$. Cohen has shown that for the FR model on $\mathbf{Z}^2$ if $p = 1$ and $0 < C < 1$, then the probability of the origin being periodic is positive.[8] Combining the proof of Theorem 4 with Theorem 6, we have the following result.

Corollary. For the FM and FR models on $\mathbf{T}^2$ with $p = 1$ and for the FR model on $\mathbf{Z}^2$ with $0 < C < p = 1$ the diffusion is non-Gaussian.

The numerical simulations of Kong and Cohen[5,8] indicate that for the FM model on $\mathbf{Z}^2$ the diffusion is Gaussian for all $p > 0$. Notice that if $p = 0$, then the FM model reduces to the DM model. In the proof of Theorem 4 we show that for the DM model, orbits are periodic with positive probability. Contrasting this fact with Theorem 6, we see that the FM model undergoes a phase transition when $p \to 0$ and that the FM and the FR models on $\mathbf{T}^2$ undergo phase transitions for the case $C < 1$ when $p \to 1$ as well as for the case $p = 1$ when $C \to 1$.

The last model we consider we call the random mirror model on $\mathbf{Z}^2$ (RM). In this model a particle propagates along the bonds of the lattice. At each instant we place, with probability C_R, C_L, or $1 - C$, a right mirror, a left mirror, or no mirror at all. If we visit a lattice site more than once, then the mirrors at this location at different times are independent of each other. Analogously, we can define a random rotator model on $\mathbf{Z}^2$ (RR).

Theorem 7. For the RM and RR models on $\mathbf{Z}^2$, if $C_R = C_L > 0$, then with probability 1 orbits are recurrent.

We note that the FM (resp. FR) model for $C_R = C_L = p = 1/2$ is equivalent to the RM (resp. RR) model for $C_R = C_L = 1/2$; thus, Theorem 7 also holds for the FM and FR models for this case. Tóth[18] has shown the existence of diffusion in a class of models generalizing the RM and RR models.

3. THE DETERMINISTIC MODELS

The main idea in the proofs of Theorems 1–3 is the following. Suppose, instead of placing a particle at the origin and following its path, we pour some fluid on the origin. The fluid spreads along the lattice, but is directed (blocked) in the obvious way by mirrors (resp. rotators). For the parameter values of Theorems 1–3, percolation theory tells us that almost surely the fluid will only reach a finite number of lattice sites. Now the particle in the DM model can only reach sites that have been wetted by such a fluid. Therefore for the parameter values in the theorems it visits almost surely a finite number of vertices and thus must be periodic. For $C < 1$, paths which start in a finite percolation cluster must automatically be finite, but paths which start in infinite percolation clusters can also be (and often are) finite. Thus, the use of percolation results is far from being necessary for the particle path to be periodic.

Proof of Theorem 1. Here the idea is that the mirrors bounding a closed path themselves form a circuit surrounding the origin on a $\mathbf{Z}^2$ lattice. This idea is exploited by showing a duality to anisotropic bond percolation on $\mathbf{Z}^2$, that is, bond percolation on $\mathbf{Z}^2$ in which each horizontal edge is open with probability p_h and each vertical edge is open with probability p_v.[4] Set $p := (p_h + p_v)/2$. If $p_h > 0$, $p_v > 0$, and $p = 1/2$, then the origin is almost surely in the interior of some open circuit.[4,19,20]

There are two dual lattices to the DM model which we can use to prove the theorem. The vertices of the first lattice are $V_1 := (1/2 + n + 2m, 1/2 + n)_{n,m \in \mathbf{Z}}$ and the vertices of the second are $V_2 := (1/2 + n + 2m, -1/2 + n)_{n,m \in \mathbf{Z}}$. If we connect each of the vertices to its four nearest neighbors (in the diagonal directions), then we form square lattices $\mathscr{L}_1$ and $\mathscr{L}_2$ oriented at a 45° angle. To prove the theorem, we need only consider one of the lattices, say $\mathscr{L} := \mathscr{L}_1$. Actually, the behavior on one lattice completely determines the behavior on the other in the case $C = 1$. Now for each $z \in \Omega$ we open (keep) bonds from $\mathscr{L}$ as follows. When $z(2m + n, n) = R$ we open the vertical bond from $(1/2 + 2m + n, 1/2 + n)$ to $(-1/2 + 2m + n, -1/2 + n)$ and when $z(2m + n + 1, n) = L$ we open the horizontal bond from $(1/2 + 2m + n, 1/2 + n)$ to $(3/2 + 2m + n, -1/2 + n)$. All other bonds are closed (erased). This gives rise to a sample configuration of anisotropic percolation on this lattice with $p_h = C_L$ and $p_v = C_R$ (here we think of $y = x$ as the vertical direction and $y = -x$ as the horizontal direction). Now applying the above-mentioned result of Kesten, we have the origin is almost surely in the interior of some open circuit. The orbit of all paths from the origin are trapped inside this circuit and thus are almost surely periodic. ∎

Using the same kind of proof, Grimmett[4] demonstrated Theorem 1 in the case $C_R = C_L = 1/2$. In this case the dual bond percolation is isotropic Bernoulli percolation. If $C < 1$, then the above argument does not work. The path boundary can consist of edges of both $\mathscr{L}_1$ and $\mathscr{L}_2$. Second, the jumps from one lattice to the other one are not edges in either lattice. Nonetheless, the probability of an open circuit around the origin is a lower bound for the probability of the origin being periodic. This lower bound is monotonically increasing in C.

Proof of Theorem 2. (i) For the proof we consider site percolation on $\mathbf{Z}^*$, the lattice whose vertices are just those of $\mathbf{Z}^2$, but each vertex is connected to its eight nearest neighbors (the so-called *-neighborhood). Note that this is not a planar graph, since the diagonal edges intersect at points which are not vertices. It is known that $p_c(\mathbf{Z}^*) = 1 - p_c(\mathbf{Z}^2)$, where the critical parameters are for site percolation on $\mathbf{Z}^*$ and $\mathbf{Z}^2$, respectively.[19,20] The constant $p_c := p_c(\mathbf{Z}^2)$ is known to be strictly larger than 1/2 and strictly less than 1. The DR model on $\mathbf{Z}^2$ is related to site percolation on $\mathbf{Z}^*$ as follows. The path of the origin is periodic if it is bounded by a closed circuit (on $\mathbf{Z}^*$) of vertices having one fixed type of scatterer. Note that this is not a necessary condition for the periodicity of the origin. A closed circuit encloses the origin with probability 1 if C_R or $C_L > p_c(\mathbf{Z}^2)$.[4,19,20]

(ii) Consider the case $C = 1$. In this case the DR model is equivalent to the DM model in the following sense. Construct two maps ϕ_1, ϕ_2: $\Omega_{\mathrm{DR}} \to \Omega_{\mathrm{DM}}$ as follows:

$$\phi_1(z)(i, j) := \begin{cases} z(i, j) & \text{if } \ i + j = 0 \bmod 2 \\ \neg\, z(i, j) & \text{if } \ i + j = 1 \bmod 2 \end{cases}$$

and

$$\phi_2(z)(i, j) := \begin{cases} z(i, j) & \text{if } \ i + j = 1 \bmod 2 \\ \neg\, z(i, j) & \text{if } \ i + j = 0 \bmod 2 \end{cases}$$

where $\neg R := L$ and $\neg L := R$. Note that both of these map the DR model with arbitrary parameters C_R and C_L to the DM model with parameters 1/2 and 1/2. If a particle starts at the origin in a vertical (resp. horizontal) direction, then its DR path on the configuration z and its DM path on the configuration $\phi_2(z)$ [resp. $\phi_1(z)$] will be identical. This follows from a simple parity condition; for both the DM and DR models the particle is traveling in the the same "direction" (i.e., horizontally or vertically) as it started on odd time steps and the other "direction" on even time steps. Thus, each DR rotator is hit by the particle from at most two directions and these directions are parallel, so the DR rotator acts like a DM mirror. Part (ii) of the theorem is now a corollary to Theorem 1. ▌

Proof of Theorem 3. Without loss of generality we can assume $C_L \geqslant 1/2$. We prove the theorem by comparing with site percolation on $\mathbf{T}^2$. For each $z \in \Omega$ open a vertex in $\mathbf{T}^2$ if the scatterer there is a left scatterer. Since $C_L \geqslant 1/2$, we have that the origin is almost surely in the interior of some open circuit.[4,19,20] Back in Ω, for either the DM or DR model, the left scatterers that appear along this circuit enclose a region from which the particle cannot escape. ▌

Proof of Theorem 4. Consider the DM model on $\mathbf{Z}^2$. If $C=0$, $C_R=0$, or $C_L=0$, then $P(\mathbf{0}, t)=0$ for $t>0$ and the diffusion is non-Gaussian. In all other cases the set of periodic points must have positive measure. For example, the set of configurations with right mirrors at the lattice points $(\pm 1, 0)$ and $(0, \pm 1)$ and left mirrors at the lattice point $(0, 0)$ and $\pm(1, 1)$ has positive measure. For these configurations the particle starting at the origin and traveling in any of the four directions is periodic. Let $\sigma(z)$ be the path of a particle starting to the right and Q the set of z for which $\sigma(z)$ is periodic. Let $Q^N = \{z \in \Omega: \mathrm{per}(\sigma(z)) = N\}$. Since $Q = \bigcup Q^N$, we have that $\mu(Q^N) > 0$ for some N. Then consider $P(\mathbf{r}, t)$ for $\mathbf{r} = \mathbf{0}$ and $t = Nm$. It follows that $P(\mathbf{0}, Nm) \geqslant (1/2)\, \mu(Q^N)$. The 1/2 arises here since Q^N only takes into account particles going to the right. However, for a normal diffusion we have that the probability density $Q(\mathbf{r}, t)$ satisfies $\lim_{t \to \infty} Q(\mathbf{r}, t) = 0$ for all $\mathbf{r} \in \mathbf{Z}^2$. Thus, the diffusion is non-Gaussian. The proof for the $\mathbf{T}^2$ lattice or the DR model is analogous. ▌

Proof of Theorem 5. A nontrivial rotational symmetry class $\{f_j\}$ on $\mathbf{Z}^2$ either has scattering rules with all four left turns [i.e., $\forall i\, \exists j$ s.t. $f_j(i) = i + 1 \bmod 4$] or all four right turns [i.e., $\forall i\, \exists j$ s.t. $f_j(i) = i - 1 \bmod 4$], or with all four U-turns [i.e., $\forall i\, \exists j$ s.t. $f_j(i) = i$]. For any of these cases it is not hard to see that periodic points have positive measure and the proof of Theorem 4 can be adapted to prove the theorem.

On $\mathbf{T}^2$ a nontrivial rotational symmetry class $\{f_j\}$ has one of the following: (i) all six left rurns [i.e., $\forall i\, \exists j$ s.t. $f_j(i) = i + 1 \bmod 6$], (ii) all six right turns [i.e., $\forall i\, \exists j$ s.t. $f_j(i) = i - 1 \bmod 6$], (iii) all six 120° left turns [i.e., $\forall i\, \exists j$ s.t. $f_j(i) = i + 2 \bmod 6$], (iv) all six 120° right turns [i.e., $\forall i\, \exists j$ s.t. $f_j(i) = i - 2 \bmod 6$], or (v) all six U-turns [i.e., $\forall i\, \exists j$ s.t. $f_j(i) = i$]. Again periodic points have positive measure and we can apply the proof of Theorem 4.

On $\mathbf{Z}^n$ there is at least one two-dimensional plane $A_{m,n}$ for which a nontrivial rotational symmetry class has scattering rules with all four left, right, or U-turns. Thus, periodic orbits have positive measure and once again we can apply the proof of Theorem 4. ∎

4. THE FLIPPING MODELS

Proof of Theorem 6. First consider the FM model on $\mathbf{Z}^2$. Suppose the particle stays in a bounded region. Consider the set of lattice points O which the path hits infinitely often. Consider the set of rightmost points $\{(i_{\max}, j)\} \subset O$ and the topmost of these points $(i_{\max}, j_{\max})$. Since the particle hits this point $(i_{\max}, j_{\max})$ infinitely often, using the Borel–Cantelli lemma and the fact that $p > 0$, we can conclude that the mirror at this location must flip from right to left infinitely often. However, when the mirror is a right mirror and the particle hits it, the particle must go further to the right or further up infinitely often, a contradiction. Thus, the particle cannot stay in any bounded region. It clearly follows that the particle can not be periodic.

For the FR model on $\mathbf{Z}^2$ if $p < 1$, again consider the rightmost point the particle hits infinitely often. Consider the events that the particle approaches this vertex on the unique horizontal edge coming from the left. The rotator at this site must be a right rotator to keep the particle in the bounded region. This contradicts the stochasticity of the rotator flipping. If the particle approaches this vertex only a finite number of times horizontally, then it approaches vertically infinitely often and a similar argument works.

Next consider the FR model on $\mathbf{Z}^2$ for $p = 1$, $C = 1$. Notice that for $C = 1$ the particle is always moving in the horizontal direction on even time steps and the vertical direction on odd time steps, or vice versa, depending only on which direction it started. Furthermore, when leaving any vertex the particle must take an even number of steps before it can return to the same vertex. Again consider the top right-hand-most point a particle hits infinitely often. The simple parity check described above tells us that the particle must always approach this vertex in the same direction, an immediate contradiction.

Finally, for the case $p = 1$ and $0 < C < 1$ for the FR model on $\mathbf{Z}^2$ a periodic orbit was constructed by Cohen.[8]

Next consider $\mathbf{T}^2$ and the FM (resp. FR) model with $p < 1$. Consider the set O of lattice points which the particle hits infinitely often and the set of topmost points $\{(i, j_{\max})\} \subset O$. Consider the rightmost point $(i_{\max}, j_{\max})$

of this set. If the particle approaches it infinitely often from below-right, then by stochasticity it must go to the right-horizontally sometimes, a contradiction. If the particle approaches infinitely often from the left-horizontally, then by stochasticity it must go up-left sometimes, a contradiction. If the particle approaches infinitely often from below-left, then by stochasticity it must go infinitely often left-horizontally, and then using the stochasticity of the first vertex with a scatterer that the particle reaches in this direction, we conclude that it must go up-right, a contradiction.

Next, for the FM and FR models on $\mathbf{T}^2$, if $p=1$ and $C<1$, we construct a periodic orbit by placing left mirrors (resp. rotators) along a hexagon and no mirror in the middle. Then any orbit starting from the origin is periodic.

Finally, the FM model on $\mathbf{T}^2$ in the case $p=1$ and $C=1$ was treated by Kong and Cohen.[6] The mechanism they describe which sends the particle to infinity in a "linear" way also works for the FR model on $\mathbf{T}^2$. ∎

5. THE RANDOM MIRROR MODEL

Proof of Theorem 7. This model is the only Markovian model we discuss. We will show that if we consider the process only at the times when it hits every second mirror, then it is actually a random walk model. More specifically, for a given realization σ of our process, construct the vectors $\{\mathbf{v}_i\}$ given by: $\mathbf{v}_0$ is the vector pointing from the origin to the second scatterer which σ hits and $\mathbf{v}_i$ is the vector pointing from the $2i$th scatterer which σ hits to the $2(i+1)$th scatterer which σ hits. A sequence $\{\mathbf{v}_i\}$ corresponds to two different paths σ. However, if we know the direction of σ at time 0, then $\{\mathbf{v}_i\}$ uniquely determines σ. The covering process $\{\mathbf{v}_i\}$ is an independent one, and thus it is a two-dimensional random walk.[21] Since $C_R=C_L=q>0$, we have that the mean of this random walk is zero. Let $|\mathbf{v}|$ denote the length of the vector $\mathbf{v}$ (for ease of computation we use $|\mathbf{v}|=|\mathbf{v}_x|+|\mathbf{v}_y|$, which is equivalent to Euclidean length) and $P(\mathbf{v})$ denote the probability of the vector $\mathbf{v}$. Let $m_2=\sum |\mathbf{v}|^2 P(\mathbf{v})$ be the second moment of our process. It is well known that two-dimensional random walks with zero mean and finite second moment are recurrent (see, for example, Spitzer[21]). To see that our random walk has finite second moment, notice that the cardinality of the set of vectors with length n is $4(n-1)$ and for any such vector $P(\mathbf{v})=q^2(1-2q)^{n-2}$. Thus,

$$\begin{aligned} m_2=\sum |\mathbf{v}|^2 P(\mathbf{v}) &= \sum_{n\geq 2} \sum_{\{\mathbf{v}:\, |\mathbf{v}|=n\}} (n)^2 P(\mathbf{v}) \\ &= 4q^2 \sum_{n\geq 2} (n-1)(n)^2 (1-2q)^{n-2} \end{aligned} \tag{3}$$

is finite. We have shown that the process $\{\mathbf{v}_i\}$ is recurrent. Since the random walk is a two-to-one cover of the RM (resp. RR) model, it follows that the RM (resp. RR) process is also recurrent. ∎

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